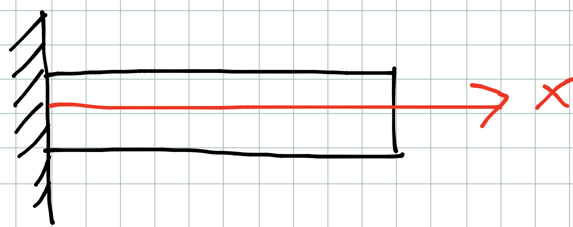
Consider a homogeneous beam undergoing torsion.



$$\Theta(x,t) = X(x)Y(t)$$

The beam has a cantilevered end on the left and a free end on the right. Assuming that the beam undergoes elastic torsion, the beam can be assumed to have the elastic response:

$$X(x) = A\sin(\alpha x) + B\cos(\alpha x) \qquad (1)$$

Where $X(x)$ is the torsional response. α is the constant arising from the second-order wave equation separation of variables. There are two boundary conditions, one at each end of the beam: $\Theta(0,t) = X(0)Y(t) = 0$

$\therefore X(0) = 0$ and similarly $X'(l) = 0$ (2)

That is, there can be no deflection at the cantilevered end and no torque at the right end. If these two boundary conditions are applied to the beam, the equation for the deflection becomes:

$$X(0) = A\sin(0) + B\cos(0) = 0$$

$$A(0) + B(1) = 0$$

$$\therefore B = 0$$

$$X'(l) = \alpha A\cos(\alpha l) = 0 \qquad (3)$$

Since the boundary condition at x=0 indicates that the second term is zero (since B=0) then only the first term needs to be used for the boundary condition at the right end of the beam. In solving equation (3), there are three options. First, A can be zero. However in that case, then equation (1) is also zero and there is no torsional deflection. The second instance is if alpha, α , is zero. But the separation of variables approach indicates that alpha must be a nonzero number if a solůtion of the form equation (1) is to be valid. So this solution is not valid. Finally, the term cos(αl) is equal to zero. If this is the case then there are an infinite number of solutions:

$$\alpha_i l = \frac{2i-1}{2} \qquad i = 1, 2, 3 \ldots \infty$$

Therefore, equation (1) for these two boundary conditions becomes

$$\theta_i(x) = A_i \sin\left(\frac{(2i-1)x}{2l}\right) \qquad (4)$$

This equation describes the deflection of the beam. If the equation is nondimensionalized by dividing both sides by the constant A, then the equation becomes the mode shape equation for the beam:

$$\frac{\theta(x)}{A_i} = \frac{A_i}{A_i}\sin\left(\frac{(2i-1)x}{2l}\right)$$

$$\phi_i(x) = \sin\left(\frac{(2i-1)x}{2l}\right) \qquad (5)$$

The mode shape has a maximum absolute deflection of one. The mode shape describes the spatial deflection of the beam if the beam can deform elastically. It is possible that the beam could move so that the torsional deflection across the entire beam is constant. In this case, the beam would simply be rotating as a single rigid body. The possibility of this behavior can also be determined mathematically. The original wave equation was separated so that

$$\frac{X''}{X} = \frac{\ddot{Y}}{Y}\frac{GJ}{I_p} = -\alpha^2 \qquad (6)$$

Previously alpha was nonzero and equation (1) was obtained. Now let alpha =0. In this instance the spatial (left hand side) portion of the equation will become:

$$\frac{X''}{X} = 0$$

Or

$$X'' = 0$$

This simple differential equation can be integrated twice to find the solution:

$$X(x) = Ax + B \qquad (7)$$

Once again the two boundary conditions must be applied to find the two constants of integration:

$$X(0) = 0 = A(0) + B$$
$$\therefore \quad B = 0$$

$$X'(l) = A = 0$$

Here, the two constants of integration are zero indicating that there is no motion as a rigid body. Hence there will be no rigid body modes.

This procedure has resolved only the spatial behavior go the beam in torsion. In order to determine the behavior of the beam at any instant in time, the differential equation that resolves the temporal or time-dependent behavior must also be resolved and combined with the spatial response of the beam:

$$\theta(x,t) = X(x)\,Y(t)$$
$$\theta(x,t) = \sum_{i=1}^{\infty} \phi(x)\,\xi(t)$$

Thus another set of time-based boundary conditions or initial conditions must be solved. This is illustrated in another example.