Consider the homogeneous beam in torsion where the right end is free and the left end is cantilevered or fixed. In the prior example the mode shapes were computed. The natural frequencies and the time dependent behavior must also be analyzed.

The mode shapes will remain the same as long as the boundary conditions remain the same.The behavior of the beam in time will change with different initial conditions.

To illustrate, if the initial conditions are those of equilibrium (no deflection and no velocity) then the beam will remain in equilibrium (Newton's first law). If they are not zero, then there will be a response.

Let the initial deflection be

$$\theta(x,0) = \sin(\pi x / 2\ell)$$

And the initial velocity to be

$$\dot{\theta}(x,0) = \sin\left(5\pi x / 2\ell\right)$$

Using the small angle and linear assumptions, the superposition of the mode shapes and generalized coordinates (time-dependent motion) will give thebeam deflection at any point in time or space:

$$\Theta(x,t) = \sum_{i=1}^{\infty} \Phi_i \xi_i$$

$$\Theta(x,t) = \sum_{i=1}^{\infty} \sin\left(\frac{\pi(2i-1)x}{2\ell}\right)\left(E_i \sin(\omega_i t) + F_i \cos(\omega_i t)\right)$$

Where $\omega_i = \alpha_i \sqrt{\frac{GJ}{I_\rho}}$. Recall that separation of variables permits the solution of the space and time variables independently:

$$\theta(x,t) = X(x)\,Y(t)$$

Therefore the initial conditions will become

$$\Theta(x,0) = \sin\left(\frac{\pi x}{2\ell}\right) = \sum_{i=1}^{\infty} \left[E_i \sin(\omega_i 0) + F_i \cos(\omega_i 0)\right] \sin\left(\frac{\pi(2i-1)x}{2\ell}\right)$$

$$\sin\left(\frac{\pi x}{2\ell}\right) = \sum_{i=1}^{\infty} F_i \sin\left(\pi\frac{(2i-1)x}{2\ell}\right) \qquad (1)$$

There is an infinite summation on the right hand side of the equation, so to ensure that the solution is valid at all i values, orthogonality is applied. Recall that orthogonality has the form:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^{\ell} \phi_i \phi_j \, dx$$

Thus, eq. (1) must be modified so that it has this form. The right hand side of the equation is close, so if the equation is multiplied by

$$\sum_{j=1}^{\infty} \phi_j$$

and then integrated over the length of the beam, the equation becomes

$$\sum_{j=1}^{\infty} \int_0^{\ell} \sin\left(\frac{\pi x}{2\ell}\right) \sin\left(\pi\frac{(2j-1)x}{2\ell}\right) dx = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left[\int_0^{\ell} \sin\left(\pi\frac{(2i-1)x}{2\ell}\right) \sin\left(\pi\frac{(2j-1)x}{2\ell}\right) dx\right] F_i$$

Consider first the right hand side of the equation (let $a_i = (2i-1)/2\ell$)

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} F_i \int_0^{\ell} \sin(a_i x)\sin(a_j x)\,dx = \begin{cases} 0 & \text{when } i \neq j \\ F_i\,\ell/2 & \text{when } i = j \end{cases}$$

Now the left hand side of the equation is analyzed:

$$\sum_{j=1}^{\infty} \int_0^{\ell} \sin\left(\frac{\pi x}{\ell}\right) \sin(a_j x)\,dx = \sum_{j=1}^{\infty} \int_0^{\ell} \sin(a_1 x)\sin(a_j x)\,dx$$

While this expression does not have an i term, the initial deflection is actually the i=1 term. When this is integrated:

$$\sum_{j=1} \int_0 \sin(a_1 x)\sin(a_j x) = \begin{cases} \ell/2 & j=1 \end{cases}$$

When both sides are compared:

$$\left.\begin{matrix} 0 & j \neq 1 \\ \ell/2 & j = 1 \end{matrix}\right\} = \begin{cases} 0 & i \neq j \\ F_i \ell/2 & i = j \end{cases}$$

So when i=j=1

$$\frac{\ell}{2} = F_1 \frac{\ell}{2}$$

$$F_1 = 1$$

Therefore the motion of the generalized coordinate is

$$\xi_i = E_i \sin(\omega_i t) + \cos(\omega_1 t)$$

It is important to note that there are currently an infinite number of terms for the cosine term but only one for the sine term. The second initial condition will now be analyzed:

$$\dot{\xi}_i = \omega_i E_i \cos(\omega_i t) - \omega_1 \sin(\omega_1 t)$$

Applying the initial condition

$$\dot{\theta}(x,t) = \sum_{i=1}^{\infty} \sin\left(\frac{\pi(2i-1)x}{2\ell}\right)\left(\omega_i E_i \cos(\omega_i t) - \omega_1 \sin(\omega_1 t)\,\delta_{i1}\right)$$

The last term has the small delta symbol that indicates that when i=1 the term is present, but for all other i values the term is zero. This is an easy way to include single terms within a series summation.

$$\dot{\theta}(x,0) = \sin\left(\frac{5\pi x}{2\ell}\right) = \sum_{i=1}^{\infty} \sin\left(\pi\frac{(2i-1)}{2\ell}x\right)\left[\omega_i E_i + 0\right]$$

Once again orthogonality must be applied to find the solution for the entire series:

$$\sum_{j=1}^{\infty} \int_0^{\ell} \sin\left(\frac{5\pi x}{2\ell}\right)\sin(a_j x)\,dx = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \omega_i E_i \int_0^{\ell} \sin(a_i x)\sin(a_j x)\,dx$$

$$\left.\begin{matrix} j = 3 & \ell/2 \\ j \neq 3 & 0 \end{matrix}\right\} = \begin{cases} \ell/2\, \omega_i E_i & i = j \\ 0 & i \neq j \end{cases}$$

$$i = j = 3 \qquad \ell/2 = \ell/2\, \omega_i E_i$$

$i = j \neq 3 \qquad 0 = \frac{\ell}{2}\omega_i E_i$

$i = j = 3 \qquad E_3 = \frac{1}{\omega_3}$

Now these two results can be combined to provide the response of the beam undergoing torsion

$$\theta(x,t) = \sin\left(\frac{\pi x}{2\ell}\right)\cos(\omega_1 t) + \sin\left(\frac{5\pi x}{2\ell}\right)\frac{\sin(\omega_3 t)}{\omega_3}$$

So from an infinite series with two unknown initial conditions the motion decomposes two a single two terms with the coefficients determined. This is usually not the case; usually there remains an infinite series, but this depends on the initial conditions and any externally applied moments.